

NAKAYAMA AUTOMORPHISM AND RIGIDITY OF DUAL REFLECTION GROUP COACTIONS

E. KIRKMAN, J. KUZMANOVICH AND J.J. ZHANG

ABSTRACT. We study homological properties and rigidity of group coactions on Artin-Schelter regular algebras.

0. INTRODUCTION

The classical Shephard-Todd-Chevalley Theorem states that if G is a finite group acting faithfully on a finite dimensional \mathbb{C} -vector space $\bigoplus_{i=1}^n \mathbb{C}x_i$, then the fixed subring $\mathbb{C}[x_1, \dots, x_n]^G$ is isomorphic to $\mathbb{C}[x_1, \dots, x_n]$ if and only if G is generated by pseudo-reflections of $\bigoplus_{i=1}^n \mathbb{C}x_i$. Such a group G is called a *reflection group*. The indecomposable complex reflection groups are classified by Shephard and Todd [ST], and there are three infinite families plus 34 exceptional groups. When the commutative polynomial rings are replaced by skew polynomial rings, one can define a notion of reflection group in this noncommutative setting. Then there is one extra class of “mystic” reflection groups $M(a, b, c)$, discovered in [KKZ2], also see [BB] for further discussion.

When G is replaced by a semisimple Hopf algebra H (which is not a group algebra), then there is no inner faithful action of H on the commutative polynomial ring by a very nice result of Etingof-Walton [EW1]. This is one of many reasons why we need to consider noncommutative Artin-Schelter regular algebras if we want to have a fruitful noncommutative invariant theory. Artin-Schelter regular algebras were introduced by Artin-Schelter [AS] in 1980’s, and by now, are considered a natural analogue of commutative polynomial rings in many respects. The definition of an Artin-Schelter regular algebra (abbreviated by AS regular) is given in Definition 1.1.

Throughout the rest of this paper, let \mathbb{k} be a base field of characteristic zero, and all vector spaces, (co)algebras, and morphisms are over \mathbb{k} . Let H be a semisimple Hopf algebra and let K be the \mathbb{k} -linear dual of H . Then K is also a semisimple Hopf algebra. It is well-known that a left H -action on an algebra A is equivalent to a right K -coaction on A , and we will use this fact freely.

In this paper we are interested in the case when H is $\mathbb{k}^G := \text{Hom}_{\mathbb{k}}(\mathbb{k}G, \mathbb{k})$, or equivalently, K is the group algebra $\mathbb{k}G$ for some finite group G . Let e denote the unit of G . As an algebra, $\mathbb{k}^G = \bigoplus_{g \in G} \mathbb{k} p_g$ where its multiplication is determined by

$$p_g p_h = \begin{cases} p_g & g = h, \\ 0 & g \neq h \end{cases}, \quad 1 = \sum_{g \in G} p_g,$$

2000 *Mathematics Subject Classification.* 16E10, 16E65, 16W30, 20J50.

Key words and phrases. Artin-Schelter regular algebra, dual reflection group, Frobenius algebra, Hilbert series, fixed subring, Nakayama automorphism, rigidity.

and its coalgebra structure is determined by

$$\Delta(p_g) = \sum_{h \in G} p_h \otimes p_{h^{-1}g}, \quad \epsilon(p_g) = \delta_{e,g} := \begin{cases} 1 & g = e, \\ 0 & g \neq e, \end{cases}$$

for all $g \in G$.

Definition 0.1. A finite group G is called a *dual reflection group* if the Hopf algebra $H := \mathbb{k}^G$ acts homogeneously and inner faithfully on a noetherian AS regular domain A generated in degree 1 such that the fixed subring A^H is again AS regular, i.e. the identity component of A under the G -grading is AS regular. In this case we say that G *coacts on A as a dual reflection group*.

An example of a dual reflection group acting on an AS regular algebras is given in Example 3.7; further examples will appear in [KKZ5]. We do not yet have a classification of dual reflection groups.

When we have a Hopf algebra (co)action on an AS regular algebra, the homological (co)determinant is defined in [KKZ3].

Definition 0.2. Suppose a finite group G coacts on A as a dual reflection group. Let D be the homological codeterminant of the $\mathbb{k}G$ -coaction on A as defined in [KKZ3, Definition 6.2] or [CWZ, Definition 1.4(b)]. Then we call the element $m := D^{-1}$ in G the *mass element* of the $\mathbb{k}G$ -coaction (or \mathbb{k}^G -action) on A .

Our first result is the following. Let $H_A(t)$ be the Hilbert series of a graded algebra A .

Theorem 0.3. *Let A be a noetherian AS regular domain generated in degree 1. Let G coact on A inner faithfully as a dual reflection group and $H = \mathbb{k}^G$. Then the following hold.*

- (1) *There is a set of homogeneous elements $\{f_g \mid g \in G\} \subseteq A$ with $f_e = 1$ such that $A = \bigoplus_{g \in G} A_g$ and $A_g = f_g \cdot A^H = A^H \cdot f_g$ for all $g \in G$.*
- (2) *There is a generating subset \mathfrak{R} of the group G satisfying $e \notin \mathfrak{R}$ such that $A_1 = \bigoplus_{g \in \mathfrak{R}} \mathbb{k}f_g \oplus (A_1 \cap A^H)$.*
- (3) *Let $l_{\mathfrak{R}}(g)$ denote the (reduced) length of $g \in G$ with respect to \mathfrak{R} . Then $\deg f_g = l_{\mathfrak{R}}(g)$ for all $g \in G$.*
- (4) *The mass element m is the unique element in G of the maximal length with respect to $l_{\mathfrak{R}}$ defined as in part (3).*
- (5) *Let $\mathbf{p}(t) = H_A(t)H_{A^H}(t)^{-1}$. Then $\mathbf{p}(t)$ is a product of cyclotomic polynomials, $\mathbf{p}(1) = |G|$ and $\deg \mathbf{p}(t) = l_{\mathfrak{R}}(m)$.*

By part (4) of the above theorem, m is also the *malth* (= maximal length) element in G . Hence a purely homologically defined invariant m has strong combinatorial flavor. Indeed this mass element will appear in several other results.

The Nakayama automorphism of an AS regular (or AS Gorenstein) algebra is an important invariant of the algebra. In the study of noetherian Hopf algebras, the explicit expression of the Nakayama automorphism has several applications in Poincaré duality [BZ, Corollary 0.4], Radford's S^4 formula [BZ, Theorem 0.6], and so on. The result about the Nakayama automorphism of a smash product [RRZ2, Theorem 0.2] partially recovers a number of previous results concerning Calabi-Yau algebras; see the discussion in [RRZ2, Introduction]. In [RRZ3], a further connection between the Nakayama automorphism of an AS regular algebra and the

Nakayama automorphism of its Ext-algebra is studied. In [LMZ], the Nakayama automorphism has been used to study the automorphism group and a cancellation problem of algebras. The Nakayama automorphism is also essential in the study of rigid dualizing complexes [BZ, VdB]. The main purpose of this paper is to describe the Nakayama automorphism and to study the homological properties of various algebras related to the dual reflection group coaction on AS regular algebras.

We will recall several definitions in Section 1. Let A be any algebra. We use μ_A to denote the Nakayama automorphism of A . If A is a connected graded algebra, then the graded Nakayama automorphism μ_A is unique (if exists) since there is no nontrivial invertible homogeneous element in A . When A is a connected graded noetherian AS Gorenstein algebra, we usually require μ_A to be the graded Nakayama automorphism of A .

From now on, let $(f_g, \mathfrak{R}, m, \mathfrak{p}(t))$ be as in Theorem 0.3. In the setting of Theorem 0.3, we define the *covariant ring* of the H -action on A to be

$$A^{\text{cov } H} := A/((A^H)_{\geq 1})$$

where $((A^H)_{\geq 1})$ is the ideal of A generated by homogeneous invariant elements in A^H of positive degree. The covariant ring is also called the coinvariant ring by some authors.

Theorem 0.4. *Under the hypothesis of Theorem 0.3.*

- (1) *The covariant ring $A^{\text{cov } H}$ is Frobenius of dimension $|G|$.*
- (2) *The Hilbert series of $A^{\text{cov } H}$ is $\mathfrak{p}(t)$. As a consequence, the coefficients of $\mathfrak{p}(t)$ are positive and the coefficient of the leading term of $\mathfrak{p}(t)$ is 1.*
- (3) *Let \bar{f}_g be the homomorphic image of f_g in $A^{\text{cov } H}$ for each $g \in G$. Then $\{\bar{f}_g \mid g \in G\}$ is a \mathbb{k} -linear basis of $A^{\text{cov } H}$ and the graded Nakayama automorphism of $A^{\text{cov } H}$ is of the form*

$$\mu_{A^{\text{cov } H}} : \bar{f}_g \rightarrow \beta(g)\bar{f}_{m g m^{-1}}, \quad \text{for all } g \in G,$$

where $\{\beta(g)\}_{g \in G}$ are some nonzero scalars in \mathbb{k} .

- (4) *The scalars in part (3) satisfy*

$$\beta(m) = 1,$$

and if m commutes with g and h and $l_{\mathfrak{R}}(gh) = l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h)$ then

$$\beta(gh) = \beta(g)\beta(h).$$

As in Theorems 0.3 and 0.4, the mass element m has special properties. We can say more next.

Theorem 0.5. *Under the hypothesis of Theorem 0.3.*

- (1) *f_m is a homogeneous normal element in A .*
- (2) *Let ϕ_m be the conjugation automorphism of A defined by*

$$a \rightarrow f_m a f_m^{-1}, \quad \text{for all } a \in A.$$

Then

$$\phi_m : f_g \rightarrow \beta(g)f_{m g m^{-1}}, \quad \text{for all } g \in G,$$

where $\beta(g)$ are defined as in Theorem 0.4(3). In other words, the graded Nakayama automorphism of the covariant ring $A^{\text{cov } H}$ is induced by the conjugation automorphism ϕ_m .

- (3) *Any \mathbb{N} -graded algebra automorphism of $A^{\text{cov } H}$ commutes with $\mu_{A^{\text{cov } H}}$.*

- (4) The automorphism ϕ_m sends A^H to A^H .

The next result tells us about the Nakayama automorphisms of other algebras related to A . The homological determinant of a Hopf algebra action is defined in [KKZ3, Definition 3.3].

Theorem 0.6. *Under the hypothesis of Theorem 0.3. Let $\eta_m = \phi_m^{-1}$.*

- (1) $\mu_{A^H} = (\eta_m \circ \mu_A)|_{A^H}$. As a consequence, μ_A sends A^H to A^H .
- (2) Three automorphisms μ_{A^H} , $\mu_A|_{A^H}$ and $\eta_m|_{A^H}$ commute with each other.
- (3) The homological determinant of the H -action on A is the projection from $H \rightarrow \mathbb{k} p_m^{-1}$.
- (4) Let trans_m^l be the left translation automorphism of $H = \mathbb{k}^G$ defined by

$$\text{trans}_m^l : p_g \rightarrow p_{mg}, \quad \text{for all } g \in G.$$

Then the Nakayama automorphism of $A \# H$ is equal to $\mu_A \# \text{trans}_m^l$.

Classically, when A is a commutative polynomial ring and G is a reflection group, the covariant ring is clearly a complete intersection, so the following is a natural question.

Question 0.7. Under the hypothesis of Theorem 0.3 (or more generally for a semisimple Hopf algebra H), is the covariant ring $A^{\text{cov } H}$ a complete intersection of GK type in the sense of [KKZ4]?

This question is still open, even for $H = \mathbb{k}^G$.

One key idea in this paper is to relate the covariant ring with the so-called *Hasse algebra* (Definition 2.3) of the group G ; this algebra has been considered before in the case that G is a Coxeter group, and was called the *nilCoxeter algebra* by Fomin-Stanley in [FS].

A secondary goal of this paper is to prove some rigidity results for group coactions on some families of AS regular algebras. A remarkable rigidity theorem of Alev-Polo [AP, Theorem 1] states: Let \mathfrak{g} and \mathfrak{g}' be two semisimple Lie algebras. Let G be a finite group of algebra automorphisms of $U(\mathfrak{g})$ such that $U(\mathfrak{g})^G \cong U(\mathfrak{g}')$. Then G is trivial and $\mathfrak{g} \cong \mathfrak{g}'$. They also proved a rigidity theorem for the Weyl algebras [AP, Theorem 2]. The authors extended Alev-Polo's rigidity theorems to the graded case [KKZ1, Theorem 0.2 and Corollary 0.4]. For group coactions, we make the following definition.

Definition 0.8. Let A be a connected graded algebra. We say that A is *rigid with respect to group coaction* if for every finite group G coacting on A homogeneously and inner faithfully, $A^{\mathbb{k}^G}$ is NOT isomorphic to A as algebras.

Our main result concerning the rigidity of the dual group action is the following result, which provides a dual version of the rigidity results proved in [KKZ1, Theorem 0.2 and Corollary 0.4]. The proof is based on the structure results Theorems 0.3 and 0.5.

Theorem 0.9. *Let \mathbb{k} be an algebraically closed field. The following AS regular algebras are rigid with respect to group coactions.*

- (1) The homogenization of the universal enveloping algebra of a finite dimensional semisimple Lie algebra $H(\mathfrak{g})$.
- (2) The Rees ring of the Weyl algebra $A_n(\mathbb{k})$ with respect to the standard filtration.

(3) *The non-PI Sklyanin algebras of global dimension at least 3.*

Remark 0.10. By using ideas and results in [EW2], one can show that there is no nontrivial semisimple Hopf algebra actions on the algebras listed in Theorem 0.9. Combining [KKZ1, Theorem 0.2 and Corollary 0.4] with Theorem 0.9, one obtains that each algebra A in Theorem 0.9 is rigid with respect to semisimple Hopf algebra actions, or, equivalently, if H is a semisimple Hopf algebra acting on A inner faithfully and $A^H \cong A$, then $H = \mathbb{k}$.

Based on the above remark, we have an immediate question:

Question 0.11. Let A be an algebra that is rigid with respect to finite group actions, see [KKZ1]. Is A rigid with respect to any semisimple Hopf algebra action?

This paper is organized as follows. We provide background material on Artin-Schelter regular algebras, the Nakayama automorphism, and local cohomology in Section 1. We study the Hasse algebra and Poincaré polynomials of a finite group in Section 2. In Section 3, we prove some basic properties concerning dual reflection groups. The results about the Nakayama automorphisms are proved in Section 4, and the proofs of Theorems 0.3, 0.4, 0.5, and 0.6 appear at the end of Section 4. Theorem 0.9, about the rigidity of the dual group action, is proved in Section 5.

1. PRELIMINARIES

An algebra A is called *connected graded* if

$$A = \mathbb{k} \oplus A_1 \oplus A_2 \oplus \cdots$$

and $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$. The Hilbert series of A is defined to be

$$H_A(t) = \sum_{i \in \mathbb{N}} (\dim_{\mathbb{k}} A_i) t^i.$$

The algebras that we use to replace commutative polynomial rings are the AS regular algebras [AS]. We recall the definition below.

Definition 1.1. A connected graded algebra A is called *Artin-Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

- (a) A has injective dimension $d < \infty$ on the left and on the right,
- (b) $\text{Ext}_A^i({}_A \mathbb{k}_A, A) = \text{Ext}_A^i(\mathbb{k}_A, A_A) = 0$ for all $i \neq d$, and
- (c) $\text{Ext}_A^d({}_A \mathbb{k}_A, A) \cong \text{Ext}_A^d(\mathbb{k}_A, A_A) \cong \mathbb{k}(\mathfrak{l})$ for some integer \mathfrak{l} . Here \mathfrak{l} is called the *AS index* of A .

If in addition,

- (d) A has finite global dimension, and
- (e) A has finite Gelfand-Kirillov dimension,

then A is called *Artin-Schelter regular* (or *AS regular*, for short) of dimension d .

Let M be an A -bimodule, and let μ, ν be algebra automorphisms of A . Then ${}^\mu M^\nu$ denotes the induced A -bimodule such that ${}^\mu M^\nu = M$ as a \mathbb{k} -space, and where

$$a * m * b = \mu(a) m \nu(b)$$

for all $a, b \in A$ and $m \in {}^\mu M^\nu (= M)$. Let 1 be the identity of A . We also use ${}^\mu M$ (respectively, M^ν) for ${}^\mu M^1$ (respectively, ${}^1 M^\nu$).

Let A be a connected graded finite dimensional algebra. We say A is a *Frobenius algebra* if there is a nondegenerate associative bilinear form

$$\langle -, - \rangle : A \times A \rightarrow \mathbb{k},$$

which is graded of degree $-\mathfrak{l}$. This is equivalent to the existence of an isomorphism $A^* \cong A(-\mathfrak{l})$ as graded left (or right) A -modules. There is a (classical) graded Nakayama automorphism $\mu \in \text{Aut}(A)$ such that $\langle a, b \rangle = \langle \mu(b), a \rangle$ for all $a, b \in A$. Further, $A^* \cong {}^\mu A^1(-\mathfrak{l})$ as graded A -bimodules. A connected graded AS Gorenstein algebra of injective dimension zero is exactly a connected graded Frobenius algebra. The Nakayama automorphism is also defined for certain classes of infinite dimensional algebras; see the next definition.

Definition 1.2. Let A be an algebra over \mathbb{k} , and let $A^e = A \otimes A^{op}$.

- (1) A is called *skew Calabi-Yau* (or *skew CY*, for short) if
 - (a) A is homologically smooth, that is, A has a projective resolution in the category $A^e\text{-Mod}$ that has finite length and such that each term in the projective resolution is finitely generated, and
 - (b) there is an integer d and an algebra automorphism μ of A such that

$$(E1.2.1) \quad \text{Ext}_{A^e}^i(A, A^e) = \begin{cases} 0 & i \neq d \\ {}^1A^\mu & i = d, \end{cases}$$

as A -bimodules, where 1 denotes the identity map of A .

- (2) If (E1.2.1) holds for some algebra automorphism μ of A , then μ is called the *Nakayama automorphism* of A , and is usually denoted by μ_A .
- (3) We call A *Calabi-Yau* (or *CY*, for short) if A is skew Calabi-Yau and μ_A is inner (or equivalently, μ_A can be chosen to be the identity map after changing the generator of the bimodule ${}^1A^\mu$).

If A is connected graded, the above definition should be made in the category of graded modules and (E1.2.1) should be replaced by

$$(E1.2.2) \quad \text{Ext}_{A^e}^i(A, A^e) = \begin{cases} 0 & i \neq d \\ {}^1A^\mu(\mathfrak{l}) & i = d, \end{cases}$$

where ${}^1A^\mu(\mathfrak{l})$ is the shift of ${}^1A^\mu$ by degree \mathfrak{l} .

We will use local cohomology later. Let A be a locally finite \mathbb{N} -graded algebra and \mathfrak{m} be the graded ideal $A_{\geq 1}$. Let $A\text{-GrMod}$ denote the category of \mathbb{Z} -graded left A -modules. For each graded left A -module M , we define

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid A_{\geq n}x = 0 \text{ for some } n \geq 1\} = \lim_{n \rightarrow \infty} \text{Hom}_A(A/A_{\geq n}, M)$$

and call this the \mathfrak{m} -torsion submodule of M . It is standard that the functor $\Gamma_{\mathfrak{m}}(-)$ is a left exact functor from $A\text{-GrMod}$ to itself. Since this category has enough injectives, the right derived functors $R^i\Gamma_{\mathfrak{m}}$ are defined and called the local cohomology functors. Explicitly, one has

$$R^i\Gamma_{\mathfrak{m}}(M) = \lim_{n \rightarrow \infty} \text{Ext}_A^i(A/A_{\geq n}, M).$$

See [AZ, VdB] for more details.

The Nakayama automorphism of a noetherian AS regular algebra can be recovered by using local cohomology [RRZ2, Lemma 3.5]:

$$(E1.2.3) \quad R^d\Gamma_{\mathfrak{m}}(A)^* \cong {}^\mu A^1(-\mathfrak{l})$$

where \mathfrak{l} is the AS index of A .

2. POINCARÉ POLYNOMIALS AND HASSE ALGEBRAS

Let G be a finite group, though some of the definitions make sense in the infinite case. Let e be the unit of G . We say $\mathfrak{R} \subseteq G$ is a set of generators of G , if \mathfrak{R} generates G and $e \notin \mathfrak{R}$ (\mathfrak{R} need not be a minimal set of generators of G). We recall some definitions.

Definition 2.1. Let \mathfrak{R} be a set of generators of a group G . The *length* of an element $g \in G$ with respect to \mathfrak{R} is defined to be

$$l_{\mathfrak{R}}(g) := \min\{n \mid v_1 \cdots v_n = g, \text{ for some } v_i \in \mathfrak{R}\}.$$

We define the length of the identity e to be 0. The length $l_{\mathfrak{R}}(g)$ is also called *reduced length* by some authors.

Definition 2.2. Let \mathfrak{R} be a set of generators of G . The *Poincaré polynomial* associated to \mathfrak{R} is defined to be

$$\mathfrak{p}_{\mathfrak{R}}(t) = \sum_{g \in G} t^{l_{\mathfrak{R}}(g)}.$$

The notion of length of a group element with respect to \mathfrak{R} , where \mathfrak{R} is the set of Coxeter generators, is standard for Coxeter groups [BjB, p.15], and its generating function is called the Poincaré polynomial in [BjB, p. 201]. There is also interest in the generating function for other groups (e.g. the alternating group [Ro, p. 3] and [BRR, p. 849]); the article [He] gives a survey of recent progress in determining the order of magnitude of the maximal length of an element of a finite group G with respect to *any* generating set of G (i.e. the maximal degree of any Poincaré polynomial of the group) for many families of linear algebraic groups and permutation groups (often with respect to $\mathfrak{R} \cup \mathfrak{R}^{-1}$).

If there is no confusion (for example, \mathfrak{R} is fixed), we might use $\mathfrak{p}(t)$ instead of $\mathfrak{p}_{\mathfrak{R}}(t)$.

Definition 2.3. Let \mathfrak{R} be a set of generators of a group G .

- (1) The *Hasse algebra* associated to \mathfrak{R} , denoted by $\mathcal{H}_G(\mathfrak{R})$, is the associated graded algebra of the group algebra $\mathbb{k}G$ with respect to the generating space $\mathbb{k}e + \mathbb{k}\mathfrak{R}$,

$$\mathcal{H}_G(\mathfrak{R}) := \bigoplus_{i=0}^{\infty} (\mathbb{k}e + \mathbb{k}\mathfrak{R})^i / (\mathbb{k}e + \mathbb{k}\mathfrak{R})^{i-1}$$

where $(\mathbb{k}e + \mathbb{k}\mathfrak{R})^0 = \mathbb{k}$ and $(\mathbb{k}e + \mathbb{k}\mathfrak{R})^{-1} = 0$.

- (1') Equivalently, the *Hasse algebra* associated to \mathfrak{R} , denoted by $\mathcal{H}_G(\mathfrak{R})$, is the associative algebra with \mathbb{k} -linear basis G together with multiplication determined by

$$g \cdot h = \begin{cases} gh & l_{\mathfrak{R}}(gh) = l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h), \\ 0 & l_{\mathfrak{R}}(gh) < l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h). \end{cases}$$

- (2) A *skew Hasse algebra* associated to \mathfrak{R} is an associative algebra with \mathbb{k} -linear basis G together with multiplication determined by

$$g \cdot h = \begin{cases} \alpha_{g,h} gh & l_{\mathfrak{R}}(gh) = l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h), \\ 0 & l_{\mathfrak{R}}(gh) < l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h), \end{cases}$$

where $\{\alpha_{g,h} \mid l_{\mathfrak{R}}(gh) = l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h)\}$ is a set of nonzero scalars in \mathbb{k} .

Remark 2.4. (1) It is easy to see that Definitions 2.3(1) and 2.3(1') of the Hasse algebra are equivalent.

- (2) The first definition can be generalized to any finite dimensional Hopf algebra K as follows: Let V be a generating \mathbb{k} -space of the associative algebra K with $1 \notin V$ that is also a right coideal of K (namely, we have $\Delta : V \rightarrow V \otimes K$). The *Hasse algebra* associated to \mathfrak{R} is defined to be

$$\mathcal{H}_K(V) := \bigoplus_{i=0}^{\infty} (\mathbb{k} + V)^i / (\mathbb{k} + V)^{i-1}.$$

On the other hand, the second definition seems easier to understand in the group case.

- (3) Following the definition, $\mathcal{H}_G(\mathfrak{R})$ is a connected graded algebra with $\deg(g) = l_{\mathfrak{R}}(g)$ for all $g \in G$.
- (4) The Poincaré polynomial $\mathfrak{p}_{\mathfrak{R}}(t)$ is the Hilbert series of the graded algebra $\mathcal{H}_G(\mathfrak{R})$.
- (5) The Hasse algebra was called the *nilCoxeter algebra* by Fomin-Stanley [FS] for Coxeter group G with Coxeter generating set \mathfrak{R} . See [Al] for some related results.

We begin with a few examples.

Example 2.5. Let G be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ with

$$ij = k = -ji, \quad i^2 = j^2 = k^2 = -1.$$

Let \mathfrak{R} be a generating set $\{x_1 := i, x_2 := j, x_3 := -j\}$. Then elements of length two are $\{y_1 := k, y_2 := -k, y_3 := -1\}$ and the unique element of length 3 is $z := -i$. The Poincaré polynomial is $\mathfrak{p}_{\mathfrak{R}}(t) = 1 + 3t + 3t^2 + t^3 = (1+t)^3$ and the Hasse algebra is $\mathcal{H} := \mathcal{H}_G(\mathfrak{R}) = \mathbb{k} \oplus \mathbb{k}x_1 \oplus \mathbb{k}x_2 \oplus \mathbb{k}x_3 \oplus \mathbb{k}y_1 \oplus \mathbb{k}y_2 \oplus \mathbb{k}y_3 \oplus \mathbb{k}z$ with multiplication determined as in Definition 2.3(1'). Explicitly, we have

multiplication	x_1	x_2	x_3	y_1	y_2	y_3	z
x_1	y_3	y_1	y_2	0	0	z	0
x_2	y_2	y_3	0	0	z	0	0
x_3	y_1	0	y_3	z	0	0	0
y_1	0	z	0	0	0	0	0
y_2	0	0	z	0	0	0	0
y_3	z	0	0	0	0	0	0
z	0	0	0	0	0	0	0

It is easy to see that this is a non-symmetric Frobenius algebra and the Nakayama automorphism of this algebra is determined by

$$\mu_{\mathcal{H}} : x_1 \mapsto x_1, \quad x_2 \mapsto x_3, \quad x_3 \mapsto x_2$$

which has order 2.

With some effort one can show that \mathcal{H} is a Koszul algebra and its Koszul dual \mathcal{E} is the AS regular algebra $\mathbb{k}\langle a_1, a_2, a_3 \rangle / (a_1a_2 + a_3a_1, a_1a_3 + a_2a_1, a_1^2 + a_2^2 + a_3^2)$. Therefore the GK-dimension of the Ext-algebra of \mathcal{H} is 3.

Example 2.6. Let G be the symmetric group S_3 and \mathfrak{R} be the generating set $\{x_1 := (12), x_2 := (23)\}$. Then elements of length two are $\{y_1 := (123), y_2 := (132)\}$

and the only element of length 3 is $z := (13)$. The Poincaré polynomial is $\mathbf{p}_{\mathfrak{R}}(t) = 1 + 2t + 2t^2 + t^3$ and the Hasse algebra is $\mathcal{H} := \mathcal{H}_G(\mathfrak{R}) = \mathbb{k} \oplus \mathbb{k}x_1 \oplus \mathbb{k}x_2 \oplus \mathbb{k}y_1 \oplus \mathbb{k}y_2 \oplus \mathbb{k}z$ with multiplication determined by

multiplication	x_1	x_2	y_1	y_2	z
x_1	0	y_1	0	z	0
x_2	y_2	0	z	0	0
y_1	z	0	0	0	0
y_2	0	z	0	0	0
z	0	0	0	0	0

This is a non-symmetric Frobenius algebra and the Nakayama automorphism of this algebra is determined by the map of interchanging x_1 with x_2 . It is easy to see that \mathcal{H} is isomorphic to $k\langle a, b \rangle / (a^2, b^2, aba - bab)$ which is not a Koszul algebra. The Ext-algebra \mathcal{E} of the \mathcal{H} , which was computed in [SV, Theorem 3.3], has GK-dimension 2.

Example 2.7. Let G be the dihedral group D_{2n} with $2n$ elements, and let $s_1 = r$ and $s_2 = r\rho$ be the usual Coxeter generators where $r^2 = \rho^n = e$ and $\rho r = r\rho^{-1}$. The Poincaré polynomial of G with this choice of generating set is

$$\mathbf{p}_{\mathfrak{R}}(t) = 1 + 2t + 2t^2 + \cdots + 2t^{n-1} + t^{n-1} = (1+t)(1+t+t^2+\cdots+t^n).$$

So the Hasse algebra has the following elements in each degree:

- degree 1: $r, r\rho$;
- degree 2: ρ^{-1}, ρ ;
- degree 3: $r\rho^{-1}, r\rho^2$;
- degree 4: ρ^{-2}, ρ^2 ; etc.

When n is odd the unique largest length element is $r\rho^{\frac{n+1}{2}}$, while when n is even it is $\rho^{\frac{n}{2}}$, which is central. Hence, in the case when n is even, $\mathcal{H}_{D_{2n}}(\mathfrak{R})$ is a symmetric Frobenius algebra, but a non-symmetric Frobenius algebra when n is odd.

Based on the above examples, here are some natural questions.

- Question 2.8.** (1) When is $\mathcal{H}_G(\mathfrak{R})$ a Frobenius algebra?
 (2) When is $\mathcal{H}_G(\mathfrak{R})$ a symmetric Frobenius algebra?
 (3) When is $\mathcal{H}_G(\mathfrak{R})$ a Koszul (or N -Koszul) algebra?
 (4) Let $\mathcal{E}_G(\mathfrak{R})$ be the Ext-algebra of the connected graded algebra $\mathcal{H}_G(\mathfrak{R})$. What is the GK-dimension of $\mathcal{E}_G(\mathfrak{R})$ in terms of (G, \mathfrak{R}) ?

We give an answer to the first question in the next theorem. A polynomial $p(t) = \sum_{i=0}^n a_i t^i$ of degree n is called *palindrome* if $a_i = a_{n-i}$ for all i . If $\mathcal{H}_G(\mathfrak{R})$ is Frobenius, then $\mathbf{p}_{\mathfrak{R}}(t)$ is palindrome. As a consequence, the leading coefficient of $\mathbf{p}_{\mathfrak{R}}(t)$ is 1.

Theorem 2.9. *Let \mathfrak{R} be a set of generators of G . Suppose that $\mathbf{p}_{\mathfrak{R}}(t)$ is palindrome. Then the following hold.*

- (1) $\mathcal{H}_G(\mathfrak{R})$ is Frobenius.
- (2) Let μ be the graded Nakayama automorphism of $\mathcal{H}_G(\mathfrak{R})$. Then μ is determined by a permutation of the elements in G of length 1.

Proof. Since $\mathbf{p}_{\mathfrak{R}}(t)$ is palindrome, the coefficient of the leading term of $\mathbf{p}_{\mathfrak{R}}(t)$ is 1. This means that there is a unique element in G of the maximal length. Let \cdot be the multiplication of $\mathcal{H}_G(\mathfrak{R})$. The group product is suppressed.

Let $m \in G$ be the unique element of the maximal length d . (Later we will see that m agrees with the mass element in Definition 0.2.) Let $\{x_{i1}, \dots, x_{in_i}\}$ be the complete list of elements in G of length i for $1 \leq i \leq d-1$. We now prove the following **claim**:

- (a) For each x_{ij} , there is a unique element of length $s := d-i$, say x_{sw} , such that $x_{ij} \cdot x_{sw} = x_{ij}x_{sw} = m$ where $x_{sw} = x_{ij}^{-1}m$ as in G .
- (b) If $t \neq w$, then $x_{ij} \cdot x_{st} = 0$.
- (c) Every element of length $d-i$ is of the form $x_{ij}^{-1}m$ (for different j).

Consequently, for any $t \neq w$ (where x_{sw} is defined as in (a)), $x_{ij} \cdot x_{st} = 0$. We prove the claim by induction on i . Initial case $i = 1$: For each fixed $j \leq n_1$, we consider $x_{1j}^{-1}m$. Since $x_{1j} \neq e$, $x_{1j}^{-1}m \neq m$. This means that $l_{\mathbb{R}}(x_{1j}^{-1}m) \leq d-1$. Since $l_{\mathbb{R}}(m) = d$ and $l_{\mathbb{R}}(x_{1j}) = 1$, $l_{\mathbb{R}}(x_{1j}^{-1}m) \geq d-1$. Thus $l_{\mathbb{R}}(x_{1j}^{-1}m) = d-1$ and $x_{1j}^{-1}m = x_{d-1w}$ for some $w \leq n_{d-1}$. We have $x_{1j} \cdot x_{d-1w} = x_{1j}x_{d-1w} = m$. For any $t \neq w$, $x_{1j}x_{d-1t} \neq m$, so $x_{1j} \cdot x_{d-1t} = 0$ by definition. Since $\mathbf{p}_{\mathbb{R}}(t)$ is palindrome, every element of length $d-1$ is of the form $x_{1j}^{-1}m$. Inductive step: Let $i \geq 1$. Suppose the claim holds for $i' \leq i$. Pick a x_{i+1j} for any fixed j , then $x_{i+1j}^{-1}m$ is not equal to any $x_{i'j'}^{-1}m$ for all $i' \leq i$. Therefore $x_{i+1j}^{-1}m$ is not of the form $x_{d-i'j'}$ for all $i' \leq i$ and all $j' \leq n_{d-i'}$. As a consequence, $l_{\mathbb{R}}(x_{i+1j}^{-1}m) \leq d-i-1$. Since $m = x_{i+1j}(x_{i+1j}^{-1}m)$, $l_{\mathbb{R}}(x_{i+1j}^{-1}m) \geq d-i-1$. Thus $l_{\mathbb{R}}(x_{i+1j}^{-1}m) = d-i-1$. Write $x_{i+1j}^{-1}m$ as x_{d-i-1w} . For any $t \neq w$, $x_{i+1j}x_{d-i-1t} \neq m$. By definition, $x_{i+1j} \cdot x_{d-i-1t} = 0$. Since $\mathbf{p}_{\mathbb{R}}(t)$ is palindrome, every element of length $d-(i+1)$ is of the form $x_{(i+1)j}^{-1}m$ for some j . Therefore we proved the claim.

- (1) Recall that, if there is a nondegenerate associative bilinear form

$$\langle -, - \rangle : A \times A \rightarrow k$$

and an algebra automorphism μ of A such that

$$\langle a, b \rangle = \langle \mu(b), a \rangle (= \langle b, \mu^{-1}(a) \rangle)$$

for all $a, b \in A$, then A is a Frobenius algebra and μ is a Nakayama automorphism of A . The Nakayama automorphism always exists for Frobenius algebras and the graded Nakayama automorphism is unique for a connected graded Frobenius algebra.

Note that $\mathcal{H}_G(\mathbb{R})$ is graded with m having the highest degree. Let $\text{pr}_{\mathbb{R}m}$ be the projection to the highest degree component of $\mathcal{H}_G(\mathbb{R})$, and define $\langle a, b \rangle = \text{pr}_{\mathbb{R}m}(ab)$. Then the **claim** implies that $\langle a, b \rangle$ is a nondegenerate associative bilinear form. Therefore $\mathcal{H}_G(\mathbb{R})$ is Frobenius.

- (2) By the **claim**, there is a permutation $\sigma \in S_{n_1}$ such that

$$(E2.9.1) \quad x_{1j} \cdot x_{d-1w} = m = x_{d-1w} \cdot x_{1\sigma(j)}$$

for all $1 \leq j \leq n_1$. Using this observation we see that μ^{-1} maps x_{1j} to $x_{1\sigma(j)}$. Since $\mathcal{H}_G(\mathbb{R})$ is generated in degree 1, μ is completely determined by σ^{-1} . \square

The permutation given in Theorem 2.9(2) is called the *Nakayama permutation* of $\mathcal{H}_G(\mathbb{R})$. In the setting of Theorem 2.9, there is the unique element in G of the maximal length with respect to the length function l . We give an answer to Question 2.8(2) next.

Theorem 2.10. *Suppose $\mathbf{p}_{\mathfrak{R}}(t)$ is palindrome. Let m be the unique element of the maximal length. Let μ be the graded Nakayama automorphism of $\mathcal{H}_G(\mathfrak{R})$.*

- (1) μ permutes the elements of G and preserves the length.
- (2) μ^{-1} equals the conjugation $\eta_m : g \rightarrow m^{-1}gm$ when applied to the basis element $g \in G$.
- (3) $\mathcal{H}_G(\mathfrak{R})$ is symmetric if and only if m is central.
- (4) Suppose $\mathbf{p}_{\mathfrak{R}}(t) = 1 + at + \sum_{i \geq 2} a_i t^i$. Then m^n is central for some n dividing $a!$.
- (5) The conjugation η_m in part (2) preserves the length of $g \in G$.

Proof. (1) By the proof of Theorem 2.9, μ^{-1} maps x_{ij} to $x_{ij'}$ where j' is determined by

$$(E2.10.1) \quad x_{ij} \cdot x_{d-iw} = x_{ij}x_{d-iw} = m = x_{d-iw}x_{ij'} = x_{d-iw} \cdot x_{ij'}.$$

Hence the assertion follows. Since $\mathcal{H}_G(\mathfrak{R})$ is generated in degree 1, this automorphism μ^{-1} is completely determined by (E2.9.1).

- (2) By part (1) and (E2.10.1), for any (i, j) ,

$$\mu^{-1}(x_{ij}) = x_{ij'} = x_{d-iw}^{-1}m = m^{-1}(mx_{d-iw}^{-1})m = m^{-1}x_{ij}m = \eta_m(x_{ij})$$

which implies the assertion.

- (3) This follows from the fact $\mu^{-1}(= \eta_m)$ is the identity if and only if m is central.

(4) Restricted to the degree 1 component of $\mathcal{H}_G(\mathfrak{R})$, η_m is a permutation in S_a . So η_m has order n where n divides $a!$. This implies that η_{m^n} is the identity, which is equivalent to the assertion that m^n is central.

- (5) This is an immediate consequence of parts (1,2). \square

When G is a Coxeter group and (G, \mathfrak{R}) a Coxeter system, the Hasse algebra $\mathcal{H}_G(\mathfrak{R})$ agrees with the *nilCoxeter algebra* of Fomin-Stanley [FS], which was studied by several people; see, for example, [Al, Ba, KM]. The Frobenius property of the nilCoxeter algebras was proved by [Al] in which some statements in Theorems 2.9 and 2.10 are proven (for the special case of nilCoxeter algebras). There are other Frobenius algebras associated to (G, \mathfrak{R}) , for example, the covariant algebra of (G, \mathfrak{R}) and the Nichols algebra (or the Nichols-Woronowicz algebra as it is called in [Ba, KM, MS]) $\mathcal{B}(V)$.

Theorem 2.11. *If the Hasse algebra $\mathcal{H}_G(\mathfrak{R})$ is Frobenius, then so is any skew Hasse algebra associated to (G, \mathfrak{R}) .*

Proof. Since $\mathcal{H}_G(\mathfrak{R})$ is Frobenius, $\mathbf{p}_{\mathfrak{R}}(t)$ is palindrome. Then all statements made in the proof of Theorem 2.9 hold.

Let A be a skew Hasse algebra associated to (G, \mathfrak{R}) . Then A is a connected graded algebra with a basis $\{g \mid g \in G\}$. Following the proof of Theorem 2.9, for each i , let $\{x_{i1}, \dots, x_{in_i}\} \subset G$ be a \mathbb{k} -linear basis of A_i , which is the complete list of elements in G of length i . Then, by the definition of skew Hasse algebra, the following hold in the algebra A :

- (1) For each x_{ij} , there is a unique element of length $s := d - i$, say x_{sw} , such that $x_{ij} \cdot x_{sw} = x_{ij}x_{sw} = \alpha m$ where $x_{sw} = x_{ij}^{-1}m$ is in G and α is a nonzero scalar dependent on x_{ij} and x_{sw} .
- (2) If $t \neq w$, then $x_{ij} \cdot x_{st} = 0$.
- (3) Every element of length $d - i$ is of the form $x_{ij}^{-1}m$ (for different j).

Let $\text{pr}_{\mathbb{k}m}$ be the projection from A to the highest degree component and define $\langle a, b \rangle = \text{pr}_{\mathbb{k}m}(ab)$. It follows from (1,2,3) that $\langle a, b \rangle$ is a nondegenerate associative bilinear form. Therefore A is Frobenius.

By arguments similar to those in the proofs of Theorems 2.9 and 2.10, we see that the graded Nakayama automorphism of A is of the form

$$(E2.11.1) \quad \mu_A : g \rightarrow \beta(g)mgm^{-1}, \quad \forall g \in G$$

where $\{\beta(g) \mid g \in G\}$ are nonzero scalars. \square

3. DUAL REFLECTION GROUPS

A commutative algebra is AS regular if and only if it is a polynomial ring, so we can extend the notion of a reflection group to groups acting on more general AS regular algebras using the following definition.

Definition 3.1. A finite group G is called a *reflection group* (in the noncommutative setting) if $\mathbb{k}G$ acts homogeneously and inner faithfully on a noetherian AS regular domain A that is generated in degree 1 such that the fixed subring A^G is AS regular.

When A is a skew polynomial ring, a special AS regular algebra, reflections (or quasi-reflections as they were called in [KKZ1, Definition 2.2]) and reflection groups were studied in [KKZ1, KKZ2]. One interesting class of reflection groups that arises in this context are the so-called mystic reflection groups [KKZ2].

More generally, we can extend the definition of reflection group to semisimple Hopf algebras in the following definition.

Definition 3.2. Let H be a semisimple Hopf algebra. We say H is a *reflection Hopf algebra* if H acts homogeneously and inner faithfully on a noetherian AS regular domain A that is generated in degree 1 such that the fixed subring A^H is again AS regular. In this case we say that H acts on A as a *reflection Hopf algebra*.

In [KKZ3, Examples 7.4 and 7.6] we show that H_8 , the Kac-Palyutkin Hopf algebra of dimension 8 (a semisimple Hopf algebra that is not isomorphic to a group algebra), is a reflection Hopf algebra for the skew polynomial algebras $\mathbb{k}_{-i}[u, v]$ and $\mathbb{k}_{-1}[u, v]$. Further results on the structure and properties of reflection Hopf algebras are the object of research in progress.

Comparing Definition 3.1 with Definition 3.2, it is clear that a reflection group is a special case of a reflection Hopf algebra. The main object of this paper is the *dual reflection group* in Definition 0.1, which is another special case of Definition 3.2. Let G be a finite group and let H be the dual Hopf algebra $\mathbb{k}^G := (\mathbb{k}G)^\circ$. We say G is a *dual reflection group* if H acts homogeneously and inner faithfully on a noetherian AS regular domain A that is generated in degree 1 such that the fixed subring A^H is AS regular [Definition 0.1]. In this case we say G coacts on A as a *dual reflection group*. The group-theoretic or combinatorial connection between reflection groups and dual reflection groups is not yet evident; Example 3.7 is one example of a dual reflection group, and [KKZ5] will contain further examples.

We note that the study of \mathbb{k}^G -actions on noncommutative algebras is an important and interesting topic related to different areas of mathematics even if G is not a dual reflection group and A is not an AS regular algebra [CM].

The algebraic and coalgebraic structure of \mathbb{k}^G were reviewed in introduction.

When A is G -graded, we can write A as $\bigoplus_{g \in G} A_g$. The following lemma is well-known.

Lemma 3.3. *Let A be a noetherian AS regular domain and H be a semisimple Hopf algebra acting homogeneously and inner faithfully on A .*

- (1) [KKZ3, Lemma 2.4] *The fixed subring A^H is noetherian and A is finitely generated over A^H on the left and on the right.*
- (2) *Suppose that the fixed subring A^H is AS regular. Then $\text{gldim } A^H = \text{gldim } A$ and A is free over A^H on both sides.*

Proof. (2) The statement of [KKZ1, Lemma 1.10(d)] is for $H = \mathbb{k}G$, but the proof of [KKZ1, Lemma 1.10(d)] works for any semisimple Hopf algebra H . \square

We need another general result.

Lemma 3.4. *Let A be a noetherian AS Gorenstein algebra and B be a subalgebra of A . Assume that*

- (i) *B is AS Gorenstein.*
- (ii) *A is finitely generated and free over B on both sides.*

Then

- (1) $\text{injdim } A = \text{injdim } B$.

Suppose further that

- (iii) $AB_{\geq 1} = B_{\geq 1}A$.

Let $C = A/I$ where I is the 2-sided ideal $AB_{\geq 1}$. Then

- (2) *C is Frobenius.*
- (3) $\mathfrak{l}_C = \mathfrak{l}_A - \mathfrak{l}_B < 0$.
- (4) $\mu(I) = I$ where μ is the Nakayama automorphism of A .

Proof. By the faithful flatness, (ii) implies that B is noetherian.

(1) Let μ and ν be the Nakayama automorphisms of A and B respectively. Let d_A and d_B be the injective dimensions of A and B , respectively (and \mathfrak{l}_A and \mathfrak{l}_B be the AS indices) as given in Definition 1.1. By [RRZ2, Lemma 3.5], ${}^\mu A^1[d_A](-\mathfrak{l}_A)$ and ${}^\nu B^1[d_B](-\mathfrak{l}_B)$ are rigid dualizing complexes over A and B , respectively. By [YZ, Theorem 3.2(i) and Proposition 3.9(i)]

$${}^\mu A^1[d_A](-\mathfrak{l}_A) \cong \text{RHom}_B(A, {}^\nu B^1[d_B](-\mathfrak{l}_B))$$

as (A, B) -bimodule complexes. Since A is free as a left B -module (and as a right B -module), the homology of $\text{RHom}_B(A, {}^\nu B^1[d_B](-\mathfrak{l}_B))$ is concentrated in complex degree $-d_B$. As a consequence, $d_A = d_B$ (that is part (1)) and

$$(E3.4.1) \quad {}^\mu A^1(-\mathfrak{l}) \cong \text{Hom}_B(A, {}^\nu B^1)$$

where $\mathfrak{l} := \mathfrak{l}_A - \mathfrak{l}_B$.

Let $C = A/B_{\geq 1}A$, which is a graded factor ring of A if $B_{\geq 1}A = AB_{\geq 1}$. Let $D = A/AB_{\geq 1}$. Then $A \otimes_B \mathbb{k} \cong D$ as left A -module and $\mathbb{k} \otimes_B A \cong C$ as a right A -module.

Since A is a free left B -module, $\text{Hom}_B(A, {}^\nu B^1)$ is a free right B -module. Then we have quasi-isomorphisms of complexes of left A -modules,

$$\begin{aligned} \text{Hom}_B(A, {}^\nu B^1) \otimes_B \mathbb{k} &\cong \text{RHom}_B(A, {}^\nu B^1) \otimes_B^L \mathbb{k} \\ &\cong \text{RHom}_B(A, {}^\nu B^1 \otimes_B^L \mathbb{k}) \\ &\cong \text{RHom}_B(A, \mathbb{k}) \cong \text{Hom}_B(A, \mathbb{k}) \\ &\cong \text{Hom}_B(A, \text{Hom}_{\mathbb{k}}(\mathbb{k} \otimes_B \mathbb{k}, \mathbb{k})) \\ &\cong \text{Hom}_{\mathbb{k}}(\mathbb{k} \otimes_B A, \mathbb{k}) \cong \text{Hom}_{\mathbb{k}}(C, \mathbb{k}) \\ &= C^*, \end{aligned}$$

where the second \cong follows from the fact that ${}_B A$ is finitely generated free over B .

(2,3,4) Now we assume that $AB_{\geq 1} = B_{\geq 1}A =: I$ is a 2-sided ideal of A . In this case $C = D$. Since I is a 2-sided ideal of A , so is $\mu^{-1}(I)$. Then

$${}^\mu A^1 \otimes_B \mathbb{k} \cong {}^\mu (A/I)^1 \cong A/(\mu^{-1}(I))$$

as left A -modules. By applying $-\otimes_B \mathbb{k}$ to (E3.4.1), the above computation shows that

$$A/(\mu^{-1}(I))(-1) \cong {}^\mu A^1(-1) \otimes_B \mathbb{k} \cong \text{Hom}_B(A, {}^\nu B^1) \otimes_B \mathbb{k} \cong C^*$$

as left A -modules. As a consequence, $\mu^{-1}(I) = I$ (which is part (4)) and

$$(E3.4.2) \quad C(-1) \cong C^*$$

as left C -modules. Now (E3.4.2) implies that C is Frobenius and $\mathfrak{l}_C = \mathfrak{l} = \mathfrak{l}_A - \mathfrak{l}_B$. Therefore parts (2,3) follow. \square

When $H = \mathbb{k}^G$ acts on A , then the fixed subring A^H is the group graded e -component of A , denoted by A_e .

Theorem 3.5. *Let A be a noetherian AS regular domain. In parts (3-5) we further assume that A is generated in degree 1. Suppose G coacts on A as a dual reflection group. Let $H = \mathbb{k}^G$.*

- (1) *There is a set of homogeneous elements $\{f_g \mid g \in G\} \subseteq A$ with $f_e = 1$ such that $A_g = f_g \cdot A_e = A_e \cdot f_g$ for all $g \in G$. As a consequence, the nonzero component of A_g with lowest degree has dimension 1.*
- (2) *$I := A(A_e)_{\geq 1}$ is a two-sided ideal and $A/I \cong \bigoplus_{g \in G} \mathbb{k} \overline{f_g}$. As a consequence, the covariant ring $A^{\text{cov } H}$ is Frobenius.*
- (3) *Suppose that, as an H -module, $A_1 \cong \bigoplus_{g \in G} (\mathbb{k} p_g)^{n_g}$ where $n_g \geq 0$. If $n_g > 0$ and $g \neq e$, then $n_g = 1$.*
- (4) *The set $\mathfrak{R} := \{g \in G \mid n_g > 0, g \neq e\}$ generates G .*
- (5) *$A^{\text{cov } H}$ is a skew Hasse algebra associated to the generating set \mathfrak{R} . As a consequence, $A^{\text{cov } H}$ is a Frobenius algebra generated in degree 1. Furthermore, the Hilbert series of $A^{\text{cov } H}$ is palindrome and is a product of cyclotomic polynomials.*

By part (1), $A = \bigoplus_{g \in G} A_e f_g$, but f_g is not invertible except for $g = e$. In some sense A is a twisted semi-group ring without the basis elements f_g being invertible. Both part (2) and (5) of the above theorem prove that $A^{\text{cov } H}$ is a Frobenius algebra generated in degree 1.

Proof of Theorem 3.5. (1) Note that $A = \bigoplus_{g \in G} A_g$ and each A_g is an A_e -bimodule. Since A is a domain and the \mathbb{k}^G -action on A is inner faithful, we obtain that $A_g \neq 0$ for all $g \in G$. By Theorem 3.3(2), A is free over A_e . Hence each A_g is free over A_e . Let a be a nonzero element in $A_{g^{-1}}$, then $aA_g \subseteq A_e$. Hence A_g has rank one as a right A_e -module, which implies that $A_g = f_g \cdot A_e$ for some homogeneous element f_g . Therefore the lowest degree of the nonzero component of A_g is $\deg f_g$. By symmetry, $A_g = A_e \cdot f_g$. The consequence is clear.

(2) By part (1), $f_g A_e = A_e f_g$. Hence $f_g(A_e)_{\geq 1} = (A_e)_{\geq 1} f_g$. Therefore

$$I = A(A_e)_{\geq 1} = \bigoplus_{g \in G} f_g(A_e)_{\geq 1} = \bigoplus_{g \in G} (A_e)_{\geq 1} f_g = (A_e)_{\geq 1} A$$

which shows that I is a 2-sided ideal. It is clear that $A/I = \bigoplus_{g \in G} \mathbb{k} \overline{f_g}$. The consequence follows from Lemma 3.4(2) since $A^{cov H} = A/I$.

(3) If $g \neq e$ and $n_g > 0$, then A_g has lowest degree 1. By part (1), $n_g = 1$.

(4) Since A/I is generated in degree 1, G is generated by $\{g \in G \mid n_g > 0, g \neq e\}$.

(5) The first assertion is easy to check and the second follows from Theorem 2.11.

Since A is free over A_e , we have $A \cong A_e \otimes A^{cov H}$ as vector spaces. Hence $H_A(t) = H_{A^e}(t)H_{A^{cov H}}(t)$. Since both $H(A)^{-1}$ and $H_{A^e}(t)^{-1}$ are products of cyclotomic polynomials, so is $H_{A^{cov H}}(t)$. It is well-known that every product of cyclotomic polynomials is palindrome. \square

Let R be any algebra and t be a normal nonzerodivisor element in R . The conjugation η_t of R is defined to be $\eta_t(r) = t^{-1}rt$ for all $r \in R$. The inverse conjugation η_t^{-1} is denoted by ϕ_t .

By Theorem 3.5(1),

$$A = \bigoplus_{g \in G} f_g B = \bigoplus_{g \in G} B f_g$$

where $B = A_e$ is AS regular. Let η_g and ϕ_g be the graded algebra automorphisms of B defined by

$$(E3.5.1) \quad \eta_g : x \rightarrow f_g^{-1} x f_g, \quad \text{and} \quad \phi_g := \eta_{f_g^{-1}} : x \rightarrow f_g x f_g^{-1}$$

for all $x \in B$. Then we have

$$f_g x = \phi_g(x) f_g$$

for all $x \in B$. Define $c_{g,h} \in B$ such that

$$(E3.5.2) \quad f_g f_h = c_{g,h} f_{gh}$$

for all $g, h \in G$. Then we have

$$\phi_g \phi_h = \phi_{f_g} \phi_{f_h} = \phi_{f_g f_h} = \phi_{c_{g,h}} \phi_{gh}$$

where $\phi_{c_{g,h}}$ is defined by sending $x \rightarrow c_{g,h} x c_{g,h}^{-1}$. If $l_{\mathbb{R}}(gh) = l_{\mathbb{R}}(g) + l_{\mathbb{R}}(h)$, then $c_{g,h}$ is a scalar and $\phi_{c_{g,h}}$ is the identity, whence $\phi_g \phi_h = \phi_{gh}$.

Then associativity shows the following.

Lemma 3.6. *Retain the above notation.*

- (1) $c_{g,h}$ is a normal element in B .
- (2) The following holds

$$c_{g,h} \phi_{gh}(b) c_{gh,k} = \phi_g \circ \phi_h(b) \phi_g(c_{h,k}) c_{g,hk} = \phi_{c_{g,h}} \circ \phi_{gh}(b) \phi_g(c_{h,k}) c_{g,hk}$$

for all $g, h, k \in G$ and $b \in B$.

Example 3.7. The dihedral group D_8 of order 8 is a dual reflection group. Note that $G := D_8$ is generated by r of order 2 and ρ of order 4 subject to the $r\rho = \rho^3r$. Let A be generated by x, y, z subject to the relations

$$\begin{aligned} zx &= qxz, \\ yx &= azy, \\ yz &= xy. \end{aligned}$$

for any $a \in \mathbb{k}^\times$ and $q^2 = 1$. It is easy to see that $A = \mathbb{k}_q[x, z][y; \sigma]$ where σ sends z to x and x to az . Hence A is an AS regular algebra of global dimension 3 that is not PI (for generic a). Define the G -degree of the generators of A as

$$\deg_G(x) = r, \deg_G(y) = r\rho, \deg_G(z) = r\rho^2.$$

It is easy to check that the defining relations of A are G -homogeneous. Therefore G coacts on A homogeneously and inner faithfully. Since $r^2 = e$, x^2, y^2, z^2 are in the fixed subring A_e . By using the PBW basis $\{x^i z^j y^k \mid i, j, k \geq 0\}$ of A , one can easily check that $x^i z^j y^k \in A_e$ if and only if all i, j, k are even. Another straightforward computation shows that $A_e = \mathbb{k}[x^2, z^2][y^2, \tau]$ where $\tau = \sigma^2|_{\mathbb{k}[x^2, z^2]}$. Therefore A_e is AS regular and G is a dual reflection group. Although other examples of dual reflection groups will appear in [KKZ5] we do not know another n with D_{2n} a dual reflection group.

Example 3.8. Let G be the quaternion group of order 8. We claim that G is not a dual reflection group.

Suppose to the contrary that G coacts on a noetherian AS regular algebra A generated in degree 1 as a dual reflection group. Then, by Theorem 3.5(5), $A^{cov H}$ is a Frobenius skew Hasse algebra, and its Hilbert series is a product of cyclotomic polynomials and palindrome.

Any generating set \mathfrak{R} of G must contain two elements of order 4, so the degree of $p_{\mathfrak{R}}(t)$ is less than or equal to 3. Since $p_{\mathfrak{R}}(1) = 8$ the only possibility is

$$p_{\mathfrak{R}}(t) = (1+t)^3 = 1 + 3t + 3t^2 + t^3.$$

Hence there must be 3 generators, say x_1, x_2, x_3 , of G -degree not equal to e . Without loss of generality (and up to a conjugation and a permutation), the only possibility is $x_1 = i, x_2 = j$ and $x_3 = -j$. The Hasse algebra with respect to this \mathfrak{R} is given in Example 2.5. Let $B = A^{\mathbb{k}^G}$. Then as a B -module we have

$$A = B \oplus x_1 B \oplus x_2 B \oplus x_3 B \oplus x_1 x_2 B \oplus x_1 x_3 B \oplus x_1^2 B \oplus x_1^3 B$$

$$\deg_G(f_g): \quad e \quad i \quad j \quad -j \quad k \quad -k \quad -1 \quad -i$$

where the second line is the G -degree of the generator of each component. Since $ji = i(-j)$, we obtain a relation

$$x_2 x_1 = a x_1 x_3$$

for some nonzero scalar $a \in \mathbb{k}$. Similarly, the following relations are forced:

$$\begin{aligned} x_3 x_1 &= b x_1 x_2, \\ x_2^2 &= c x_1^2, \\ x_3^2 &= d x_1^2, \end{aligned}$$

for scalars $b, c, d \in \mathbb{k}^\times$.

We can rescale x_2 to take $c = 1$ (giving possibly new a, b, d) and rescale x_3 to make $d = 1$ (possibly changing a, b). Since x_1^2 commutes with x_2 and x_3 , $a^2 = b^2 = 1$ and x_2x_3 and x_3x_2 commute with each other. Also $x_3^2 = x_3x_1^2 = bx_1x_2x_1 = abx_1^2x_3 = abx_3^2$ so $ab = 1$. Combining with $a^2 = b^2 = 1$, we obtain that either $a = b = 1$ or $a = b = -1$. There could be other generators y_j s of G -degree e and the other relations involving x_i s and y_j s, but we will get a contradiction using the subalgebra S of A generated by the x_1, x_2, x_3 .

We first claim that the Koszul dual of S is infinite dimensional. Using the comments about the scalars above, the relations in S are:

$$\begin{aligned} x_2x_1 \pm x_1x_3 &= 0, \\ x_3x_1 \pm x_1x_2 &= 0, \\ x_2^2 - x_1^2 &= 0, \\ x_3^2 - x_1^2 &= 0. \end{aligned}$$

Then Koszul dual (also called the quadratic dual) $S^!$ of S is generated by $\widehat{x_1}, \widehat{x_2}, \widehat{x_3}$ with relations:

$$\begin{aligned} \widehat{x_2x_1} \mp \widehat{x_1x_3} &= 0, \\ \widehat{x_3x_1} \mp \widehat{x_1x_2} &= 0, \\ \widehat{x_1}^2 + \widehat{x_2}^2 + \widehat{x_3}^2 &= 0, \\ \widehat{x_2x_3} &= 0, \\ \widehat{x_3x_2} &= 0. \end{aligned}$$

The quadratic algebra generated by $\widehat{x_1}, \widehat{x_2}, \widehat{x_3}$ subject to the first three relation is an AS regular algebra of global dimension 3, denoted by D . It is easy to see that $x_2x_3 + x_3x_2, x_2x_3$ is a sequence of normal elements in D . Thus

$$\text{GKdim } S^! = \text{GKdim } D / (x_2x_3 + x_3x_2, x_2x_3) = 1,$$

which implies that $S^!$ is infinite dimensional.

Next we consider the natural graded algebra map $f : S \rightarrow A$. Note that this map is injective when restricted to degree 1. By taking the Koszul dual, we have a graded algebra map

$$f^! : A^! \rightarrow S^!.$$

Since f is injective in degree 1, $f^!$ is surjective in degree 1. Since $S^!$ is generated in degree 1, $f^!$ is surjective. We have shown that $S^!$ is infinite dimensional, so is $A^!$. By [PP, Proposition 1.3.1, p. 7] $A^!$ is a subalgebra of $\text{Ext}_A^*(\mathbb{k}, \mathbb{k})$. We obtain that $\text{Ext}_A^*(\mathbb{k}, \mathbb{k})$ is infinite dimensional, a contradiction to the fact that A is AS regular.

4. NAKAYAMA AUTOMORPHISMS

For any algebra A , the Nakayama automorphism of A (if it exists) is denoted by μ_A . In this section we study the interplay between the Nakayama automorphisms of the algebras $\mathcal{H}_G(\mathbb{R})$, A , $A \# \mathbb{k}^G$, A_e and $A^{\text{cov } \mathbb{k}^G}$. We need to use some facts about the local cohomology that were reviewed in Section 1.

4.1. Nakayama automorphism of $\mathcal{H}_G(\mathbb{R})$ and skew Hasse algebras. By Theorem 2.10(2), we have

$$(E4.0.1) \quad \mu_{\mathcal{H}_G(\mathbb{R})} : g \rightarrow m g m^{-1}, \quad \forall g \in G.$$

By (E2.11.1), the Nakayama automorphism of a skew Hasse algebra B is of the form

$$(E4.0.2) \quad \mu_B : g \rightarrow \beta(g)mgm^{-1}, \quad \forall g \in G,$$

where $\{\beta(g) \mid g \in G\}$ are nonzero scalars in \mathbb{k} .

4.2. Nakayama automorphism of A_e . Let ϕ_g be defined as in (E3.5.1).

Lemma 4.1. *Let A be a noetherian AS Gorenstein algebra of injective dimension d , and let σ be a graded algebra automorphism of A . Let M be an A -bimodule.*

- (1) $R^i\Gamma_{\mathfrak{m}}(M^\sigma) = R^i\Gamma_{\mathfrak{m}}(M)^\sigma$ for all i .
- (2) $\text{Hom}_{\mathbb{k}}(M^\sigma, \mathbb{k}) = {}^\sigma \text{Hom}_{\mathbb{k}}(M, \mathbb{k})$.
- (3) Suppose B is an AS Gorenstein subalgebra of A such that there is a G -graded decomposition

$$A = \bigoplus_{g \in G} f_g \cdot B = \bigoplus_{g \in G} B \cdot f_g,$$

for some elements $\{f_g \mid g \in G\}$, where each $f_g \cdot B (= B \cdot f_g)$ is isomorphic to ${}^1B^{\phi_g}$. Then there is a $\mathbb{Z} \times G$ -graded isomorphism of B -bimodules

$$(E4.1.1) \quad {}^{\mu_A}A^1(-\mathfrak{l}_A) \cong R^d\Gamma_{\mathfrak{m}}(A)^* \cong \bigoplus_{g \in G} f_g^* \cdot \{\phi_g \mu_B B^1\}(-\mathfrak{l}_B)$$

where f_g^* is a B -central generator of bidegree $(-\deg f_g, g^{-1})$.

- (4) μ_A maps B to B and

$$\mu_A|_B = \phi_m \mu_B \quad \text{or} \quad \mu_B = (\eta_m \mu_A)|_B.$$

Proof. (1,2) Straightforward.

- (3) By (E1.2.3),

$$(E4.1.2) \quad {}^{\mu_A}A^1(-\mathfrak{l}_A) \cong R^d\Gamma_{\mathfrak{m}}(A)^*.$$

Since A is finitely generated over B on both sides, $R^i\Gamma_{\mathfrak{m}}(M) = R^i\Gamma_{\mathfrak{m}_B}(M)$ for all i and all graded A -bimodules M [AZ]. Here $\mathfrak{m}_B = B_{\geq 1}$. By Lemma 3.4(1), A and B have the same injective dimension, say d . Hence we have $\mathbb{Z} \times G$ -graded isomorphisms of B -bimodules

$$\begin{aligned} R^d\Gamma_{\mathfrak{m}}(A)^* &\cong R^d\Gamma_{\mathfrak{m}_B}(\bigoplus_{g \in G} f_g \cdot B)^* \\ &\cong \bigoplus_{g \in G} R^d\Gamma_{\mathfrak{m}_B}(f_g \cdot {}^1B^{\phi_g})^* \quad \text{viewing } f_g \text{ as a } B\text{-central generator} \\ &\cong \bigoplus_{g \in G} f_g^* \cdot R^d\Gamma_{\mathfrak{m}_B}({}^1B^{\phi_g})^* \quad \text{viewing } f_g^* \text{ as a } B\text{-central generator} \\ &\cong \bigoplus_{g \in G} f_g^* \cdot (R^d\Gamma_{\mathfrak{m}_B}(B)^{\phi_g})^* \\ &\cong \bigoplus_{g \in G} f_g^* \cdot {}^{\phi_g}(R^d\Gamma_{\mathfrak{m}_B}(B)^*) \\ &\cong \bigoplus_{g \in G} f_g^* \cdot {}^{\phi_g}({}^{\mu_B}B^1)(-\mathfrak{l}_B) \\ &\cong \bigoplus_{g \in G} f_g^* \cdot {}^{\mu_B \circ \phi_g}B^1(-\mathfrak{l}_B). \end{aligned}$$

Combining the above with (E4.1.2), we obtain (E4.1.1).

(4) Let s be the lowest \mathbb{Z} -degree element in ${}^{\mu_A}A^1(-\mathfrak{l}_A)$, which corresponds to f_m^* by (E4.1.1). For every $b \in B$, by using (E4.1.1),

$$s\mu_A(b) = b \cdot s = s(\mu_B\phi_m)(b)$$

for all $b \in B$. Then $\mu_A(b) = \mu(B)\phi_m(b)$ for all $b \in B$. The assertion follows. \square

4.3. Nakayama automorphism of $A\#\mathbb{k}^G$. Since \mathbb{k}^G is commutative, $\mu_{\mathbb{k}^G}$ is the identity. By [RRZ2, Theorem 0.2], the Nakayama automorphism of $A\#\mathbb{k}^G$ is given by

$$\mu_{A\#\mathbb{k}^G} = \mu_A\#(\mu_{\mathbb{k}^G} \circ \Xi_{\text{hdet}}^l) = \mu_A\#\Xi_{\text{hdet}}^l$$

where hdet is the homological determinant of the \mathbb{k}^G -action on A , and Ξ_{hdet}^l is the corresponding left winding automorphism. We start with a calculation of the hdet .

Proposition 4.2. *Retain the above notation.*

- (1) *The homological determinant hdet of the \mathbb{k}^G action on A is the projection of \mathbb{k}^G onto $\mathbb{k}_{p_{m-1}}$ where m is the unique maximal length element in G . As a consequence, m^{-1} is the homological codeterminant of G -coaction on A .*
- (2) $\mathfrak{l}_{A_e} = \mathfrak{l}_A + \deg \mathfrak{p}_{\mathfrak{R}}(t)$.
- (3) *The left winding automorphism Ξ_{hdet}^l is the left translation trans_m^l by m .*

Remark 4.3. Retain the notation as in Proposition 4.2.

- (1) Proposition 4.2(1) asserts that m is in fact the mass element of G -coaction on A defined in Definition 0.2.
- (2) Proposition 4.2(2) follows also from Lemma 3.4(3).

Proof of Proposition 4.2. (1,2) Let $H = \mathbb{k}^G$. By [KKZ3, Proposition 5.3(d)], hdet is determined by the following: Let u be a nonzero element in $\text{Ext}_A^d(\mathbb{k}, A)$ where d is the injective dimension of A . Then there is an induced algebra homomorphism $\eta' : H \rightarrow \mathbb{k}$ satisfying

$$h \cdot u = \eta(h)u$$

for all $h \in H$. The homological determinant hdet is equal to $\eta' \circ S$. Since $H = \mathbb{k}^G$, the η' is the projection pr_g from H to \mathbb{k}_{p_g} for some $g \in G$. In this case, we just say that η' corresponds to a group element $g \in G$. In fact, when we view A is a G -graded algebra, g is the G -degree of u . Thus $\text{hdet} = \eta' \circ S = pr_{g^{-1}}$ corresponds to the element g^{-1} .

Now we consider $\text{Ext}_{A_e}^d(\mathbb{k}, A)$. Since A is free over A_e on the left and the right, by the change of rings, there are isomorphisms of G -graded vector spaces

$$\text{Ext}_{A_e}^d(\mathbb{k}, A) \cong \text{Ext}_A^d(A \otimes_{A_e} \mathbb{k}, A) \cong \text{Ext}_A^d(A^{cov \mathbb{k}^G}, A) \cong (A^{cov \mathbb{k}^G})^* \otimes_{\mathbb{k}} \text{Ext}_A^d(\mathbb{k}, A).$$

Since $A = \bigoplus_{g \in G} f_g A_e \cong A_e \otimes A^{cov \mathbb{k}^G}$ as G -graded A_e -module, $\text{Ext}_{A_e}^d(\mathbb{k}, A) \cong \text{Ext}_{A_e}^d(\mathbb{k}, A_e) \otimes A^{cov \mathbb{k}^G}$.

By definition, $\text{Ext}_{A_e}^d(\mathbb{k}, A_e)$ has bidegree $(-\mathfrak{l}_{A_e}, e)$, while $\text{Ext}_A^d(\mathbb{k}, A)$ has bidegree $(-\mathfrak{l}_A, g)$. Since $\text{Ext}_{A_e}^d(\mathbb{k}, A) \cong \text{Ext}_{A_e}^d(\mathbb{k}, A_e) \otimes A^{cov \mathbb{k}^G}$, the lowest \mathbb{Z} -degree element in $\text{Ext}_{A_e}^d(\mathbb{k}, A)$ has bidegree $((-\mathfrak{l}_{A_e}, e)$, and the lowest \mathbb{Z} -degree element in $(A^{cov \mathbb{k}^G})^* \otimes_{\mathbb{k}} \text{Ext}_A^d(\mathbb{k}, A)$ has bidegree $(-\deg \mathfrak{p}_{\mathfrak{R}}(t), -m) + (-\mathfrak{l}_A, g)$. Therefore $g = m$ and $\mathfrak{l}_{A_e} = \mathfrak{l}_A + \deg \mathfrak{p}_{\mathfrak{R}}(t)$. Hence the assertions follow.

(3) By definition, for any $p_h \in \mathbb{k}^G$,

$$\Xi_{\text{hdet}}^l(p_h) = \sum \text{hdet}((p_h)_1)(p_h)_2 = \sum \text{hdet}(p_s)p_{s^{-1}h} = \text{hdet}(p_{m^{-1}})p_{mh} = p_{mh}$$

which is the left translation by m . \square

We have an immediate consequence.

Corollary 4.4. *The Nakayama automorphism of the $A \# \mathbb{k}^G$ is given by $\mu_{A \# \mathbb{k}^G} = \mu_A \# \text{trans}_m^l$.*

Proof. Since \mathbb{k}^G is semisimple, S^2 is the identity. Then assertion follows from [RRZ2, Theorem 0.2] and Proposition 4.2. \square

4.4. Nakayama automorphism of $A^{\text{cov } \mathbb{k}^G}$. We have an exact sequence of graded algebras

$$1 \rightarrow A_e \rightarrow A \rightarrow A^{\text{cov } \mathbb{k}^G} \rightarrow 1.$$

Also we can describe $A^{\text{cov } \mathbb{k}^G}$ as a skew Hasse algebra, with \mathbb{k} -linear basis $\{\overline{f_g} \mid g \in G\}$, and the multiplication of $A^{\text{cov } \mathbb{k}^G}$ is determined by

$$\overline{f_g} \cdot \overline{f_h} = \begin{cases} \alpha(g, h) \overline{f_{gh}} & l_{\mathfrak{R}}(gh) = l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h), \\ 0 & l_{\mathfrak{R}}(gh) < l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h), \end{cases}$$

where $\{\alpha(g, h) \mid l_{\mathfrak{R}}(gh) = l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h)\}$ is a set of nonzero scalars in \mathbb{k} , see Definition 2.3(2). Since $l_{\mathfrak{R}}(g) = l_{\mathfrak{R}}(mgm^{-1})$ [Theorem 2.10] and $l_{\mathfrak{R}}(mg^{-1}) = l_{\mathfrak{R}}(m) - l_{\mathfrak{R}}(g)$ [Proof of Theorem 2.9], we have

$$\begin{aligned} \overline{f_{mg^{-1}}} \cdot \overline{f_g} &= \alpha(mg^{-1}, g) \overline{f_m} \\ &= \alpha(mg^{-1}, g) \alpha(mgm^{-1}, mg^{-1})^{-1} \overline{f_{mgm^{-1}}} \cdot \overline{f_{mg^{-1}}}. \end{aligned}$$

Combining the above with (E2.11.1), the Nakayama automorphism of $A^{\text{cov } \mathbb{k}^G}$ is

$$(E4.4.1) \quad \mu : \overline{f_g} \rightarrow \beta(g) \overline{f_{mgm^{-1}}}$$

with

$$(E4.4.2) \quad \beta(g) = \alpha(mg^{-1}, g) \alpha(mgm^{-1}, mg^{-1})^{-1}$$

for all $g \in G$.

If f_m is a normal element in A (which is a domain), then we can define the conjugation automorphism

$$(E4.4.3) \quad \phi_m : x \rightarrow f_m x f_m^{-1}, \quad \forall x \in A$$

which agrees with the form given in (E3.5.1) when restricted to the subalgebra $B = A^{\text{cov } \mathbb{k}^G}$.

Proposition 4.5. *Suppose G coacts on A as a dual reflection group.*

- (1) f_m is a normal element.
- (2) The Nakayama automorphism of $A^{\text{cov } \mathbb{k}^G}$ is equal to the induced automorphism of ϕ_m of A given in (E4.4.3).

Proof. (1) Let $B = A^{cov \mathbb{k}^G}$. If $b \in B$, then $f_m b = b' f_m$ by Theorem 3.5(1). If $x = f_g$, we have

$$\begin{aligned} f_m f_g &= \alpha(mgm^{-1}, mg^{-1})^{-1} f_{mgm^{-1}} f_{mg^{-1}} f_g \\ &= \alpha(mgm^{-1}, mg^{-1})^{-1} f_{mgm^{-1}} \alpha(mg^{-1}, g) f_m \\ &= \beta(g) f_{mgm^{-1}} f_m. \end{aligned}$$

Since A is generated by B and $\{f_g \mid g \in G\}$, f_m is normal.

(2) By the computation in the proof of part (1),

$$f_m f_g = \beta(g) f_{mgm^{-1}} f_m$$

which is equivalent to $\phi_m(f_g) = \beta(g) f_{mgm^{-1}}$ for all $g \in G$. The assertion follows by (E4.4.1). \square

We finish this section with proofs of the main results.

Proof of Theorem 0.3. (1) This is Theorem 3.5(1).

(2) This follows by combining parts (3) and (4) of Theorem 3.5.

(3) By Theorem 3.5(5), $A^{cov \mathbb{k}^G}$ is a skew Hasse algebra. Note that $\deg(f_g) = \deg(\overline{f_g})$ where $\overline{f_g}$ is the image of f_g in $A^{cov \mathbb{k}^G}$. As a G -graded vector space, any skew Hasse algebra is isomorphic to the associated Hasse algebra. The assertion follows by Remark 2.4(3).

(4) We use m to denote the element in G of the maximal length with respect to $l_{\mathfrak{R}}$. By Remark 4.3(1), m equals the mass element defined in Definition 0.2.

(5) Since A is free over A^H and $A \cong A^H \otimes A^{cov H}$ as graded vector spaces, $\mathfrak{p}(t)$ is the Hilbert series of $A^{cov H}$. Since $A^{cov \mathbb{k}^G}$ is a skew Hasse algebra, $\mathfrak{p}(1) = |G|$ and $\deg \mathfrak{p}(t) = l_{\mathfrak{R}}(m)$. By Theorem 3.5(5), $\mathfrak{p}(t)$ is palindrome and is a product of cyclotomic polynomials. \square

Proof of Theorem 0.4. (1) By Lemma 3.4(2), $C = A^{cov \mathbb{k}^G}$ is Frobenius. Since $A^{cov \mathbb{k}^G}$ is a skew Hasse algebra [Theorem 3.5(5)], $\dim A^{cov \mathbb{k}^G} = |G|$.

(2) The first assertion is proven in the proof of Theorem 0.3(5). The consequence is clear.

(3) This is (E4.4.1) or the proof of Proposition 4.5.

(4) By Proposition 4.5(2), $\mu_{A^{cov \mathbb{k}^G}}$ is induced by the conjugation $\phi_m : x \rightarrow f_m x f_m^{-1}$. Note that $\phi_m(f_m) = f_m$, which implies that $\beta(m) = 1$. For any $g, h \in G$ with $l_{\mathfrak{R}}(gh) = l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h)$, we have $f_g f_h = \alpha(g, h) f_{gh}$ for some nonzero scalar $\alpha(g, h)$. Then

$$\begin{aligned} \phi_m(f_{gh}) &= \alpha(g, h)^{-1} \phi_m(f_g f_h) \\ &= \alpha(g, h)^{-1} \phi_m(f_g) \phi_m(f_h) \\ &= \alpha(g, h)^{-1} \beta(g) \beta(h) f_{mgm^{-1}} f_{mhm^{-1}} \\ &= \alpha(g, h)^{-1} \beta(g) \beta(h) \alpha(mgm^{-1}, mhm^{-1}) f_{mghm^{-1}} \\ &= \alpha(g, h)^{-1} \alpha(mgm^{-1}, mhm^{-1}) \beta(g) \beta(h) f_{mghm^{-1}}. \end{aligned}$$

Hence $\beta(gh) = \alpha(g, h)^{-1} \alpha(mgm^{-1}, mhm^{-1}) \beta(g) \beta(h)$ for all g, h satisfying $l_{\mathfrak{R}}(gh) = l_{\mathfrak{R}}(g) + l_{\mathfrak{R}}(h)$. If m commutes with g and h , then $\beta(gh) = \beta(g) \beta(h)$. \square

Proof of Theorem 0.5. (1) This is Proposition 4.5(1).

(2) See Proposition 4.5(1) and its proof.

(3) This follows from [RRZ2, Lemma 5.3(b)].

(4) This follows from Theorem 3.5(1). \square

Proof of Theorem 0.6. (1) This is Lemma 4.1(4).

(2) By [RRZ2, Lemma 5.3(b)], $\mu_{A^{\mathbb{k}^G}}$ is in the center of the group $\text{Aut}_{gr}(A^{\mathbb{k}^G})$. The assertion follows from part (1).

(3) This is Proposition 4.2(1).

(4) This is Corollary 4.4. \square

5. RIGIDITY

In this section we prove that some families of AS regular algebras are rigid with respect to group coactions. Assume that \mathbb{k} is an algebraic closed field of characteristic zero in this section.

Lemma 5.1. *Let A be a noetherian AS regular domain generated in degree one and G be a non-trivial finite group coacting on A homogeneously and inner faithfully as a dual reflection group. Then there is a finite set of nonzero elements $\{z_1, \dots, z_w\}$ in degree one such that the product $z_1 z_2 \cdots z_w$ is a normal element in A .*

Proof. Let m be the mass element as in Definition 0.2 and let $l_{\mathfrak{R}}(m) = w$ be the length of m with respect to \mathfrak{R} as given in Theorem 0.3(2). Write $m = g_1 g_2 \cdots g_w$ where $g_i \in \mathfrak{R}$.

Let f_g be the element in A as defined in Theorem 0.3. Since $l_{\mathfrak{R}}(m) = \sum_{i=1}^w l_{\mathfrak{R}}(g_i)$, by the discussion after Theorem 3.5, (E3.5.2) and induction, one see that

$$(E5.1.1) \quad f_m = c f_{g_1} f_{g_2} \cdots f_{g_w}$$

for some nonzero scalar c . Since f_m is normal by Theorem 0.5(1), the assertion follows by setting $z_i = f_{g_i}$ for $i = 1, 2, \dots, w$. \square

Let \mathfrak{g} be a finite dimensional Lie algebra, and $U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} . Let $H(\mathfrak{g})$ be the homogenization of $U(\mathfrak{g})$. Note that $H(\mathfrak{g})$ is a connected graded algebra generated by the vector space $\mathfrak{g} \oplus \mathbb{k}t$ subject to the relations

$$at = ta, \quad \text{and} \quad ab - ba = [a, b]t$$

for all $a, b \in \mathfrak{g}$. It is well known that $H(\mathfrak{g})$ is a noetherian AS domain of global dimension $d := \dim \mathfrak{g} + 1$ and its Hilbert series is $(1 - t)^{-d}$.

The proof of part (2) of the following lemma is due to Monty McGovern. We thank him for allowing us to include his proof here.

Lemma 5.2. *Let $H(\mathfrak{g})$ be the homogenization of the universal enveloping algebra of a finite dimensional semisimple Lie algebra.*

(1) *If $f \in H(\mathfrak{g})$ is a homogeneous normal element, then f is central.*

(2) *There is no central element $f \in U(\mathfrak{g})$ such that f is a nontrivial product of elements in $(\mathbb{k} + \mathfrak{g}) \setminus \mathbb{k}$.*

Proof. (1) Let $f = t^i f_0$ where $i \geq 0$ and f_0 does not have a factor of t . Since t is central, we may assume that $f = f_0$ which is not divisible by t . For any $\ell \in \mathfrak{g}$, since f is normal and homogeneous, we have

$$(E5.2.1) \quad f(\ell) = (a(\ell)t + \ell')f$$

for some $a(\ell) \in \mathbb{k}$. Passing equation (E5.2.1) to the quotient ring $S(\mathfrak{g}) = H(\mathfrak{g})/(t)$, we obtain that

$$\bar{f}\ell = \ell'\bar{f}$$

where $\bar{f} \neq 0$ as f is not divisible by t . Since $S(\mathfrak{g})$ is commutative, $\ell = \ell'$. Thus, for every $\ell \in \mathfrak{g}$,

$$[f, \ell] = f\ell - \ell f = a(\ell)tf$$

for some $a(\ell) \in \mathbb{k}$. It is easy to check that

$$[f, [\ell_1, \ell_2]t] = (a(\ell_1)a(\ell_2) - a(\ell_2)a(\ell_1))t^2f = 0$$

for all $\ell_1, \ell_2 \in \mathfrak{g}$. Since \mathfrak{g} is semisimple, $[f, \ell] = 0$ for all $\ell \in \mathfrak{g}$. Since t is central, f commutes with all elements in degree 1. The assertion follows.

(2) Let G be the Lie group associated to \mathfrak{g} and consider the adjoint action of G on \mathfrak{g} that extends naturally to the action on both the symmetric algebra $S(\mathfrak{g})$ and the enveloping algebra $U(\mathfrak{g})$. Given any product

$$f := (a_1 + \ell_1) \cdots (a_n + \ell_n), \quad a_i \in \mathbb{k}, 0 \neq \ell_i \in \mathfrak{g},$$

of elements of $(\mathbb{k} + \mathfrak{g}) \setminus \mathbb{k}$ in $U(\mathfrak{g})$, for some $n \geq 1$, assume to the contrary that f is in the center of $U(\mathfrak{g})$. By an elementary property of the adjoint representation, we have $g(f) = f$ for all $g \in G$. By using the standard filtration, $g(f) = \bar{f}$ for all $g \in G$ when $\bar{f} := \ell_1 \cdots \ell_n$ is considered as an element in $S(\mathfrak{g})$. Now we choose g in G such that $g(\ell_1)$ is not a scalar multiple of ℓ_i for any i . This is possible since \mathfrak{g} is semisimple and G acts on \mathfrak{g} with no nonzero fixed points. By unique factorization in $S(\mathfrak{g})$, $g(\ell_1 \cdots \ell_n)$, which is $g(\ell_1) \cdots g(\ell_n)$, cannot coincide with $\ell_1 \cdots \ell_n$. Therefore $g(\bar{f}) \neq \bar{f}$, yielding a contradiction. \square

Lemma 5.3. *Let A be the homogenization of the universal enveloping algebra of a finite dimensional semisimple Lie algebra $H(\mathfrak{g})$. Let $\{z_1, \dots, z_w\} \subseteq A$ be a set of nonzero elements of degree one such that the product $z_1 z_2 \cdots z_w$ is a normal element in A . Then each z_i is a scalar multiple of t .*

Proof. Since t is central, we can remove those z_i of the form at for some $a \in \mathbb{k}$. Thus each z_i is $a_i t + \ell_i$ where $a_i \in \mathbb{k}$ and $0 \neq \ell_i \in \mathfrak{g}$. By Lemma 5.2(1), $z_1 \cdots z_n$ is central. Then $f := \pi(z_1) \cdots \pi(z_n)$ is central in $U(\mathfrak{g})$ where π is the quotient map $H(\mathfrak{g}) \rightarrow H(\mathfrak{g})/(t-1) = U(\mathfrak{g})$. By Lemma 5.2(2), f is trivial. So $n = 0$. This means that each z_i is of the form at for $a \in \mathbb{k}$. \square

Let $A_n(\mathbb{k})$ be the n th Weyl algebra generated by $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the relations

$$[x_i, x_j] = 0 = [y_i, y_j], \quad [x_i, y_j] = \delta_{ij}.$$

The Rees ring of $A_n(\mathbb{k})$ with respect to the standard filtration is generated by $x_1, \dots, x_n, y_1, \dots, y_n, t$ subject to the relations

$$[x_i, x_j] = 0 = [y_i, y_j] = [t, x_i] = [t, y_i], \quad [x_i, y_j] = \delta_{ij}t^2.$$

Lemma 5.4. *Let A be the Rees ring of the Weyl algebra $A_n(\mathbb{k})$ with respect to the standard filtration. Let $\{z_1, \dots, z_w\} \subseteq A$ be a set of nonzero elements of degree one such that the product $z_1 z_2 \cdots z_w$ is a normal element in A . Then each z_i is a scalar multiple of t .*

Proof. Up to a scalar z_i is of the form $t + f_i$ or f_i where $f_i \in V := \bigoplus_{i=1}^n (\mathbb{k}x_i + \mathbb{k}y_i)$. When $z_i = f_i$, then $f_i \neq 0$. Let $z = z_1 z_2 \cdots z_w$ and consider the algebra map $\phi : A \rightarrow A/(t-1) = A_n(\mathbb{k})$. Then $\phi(z_i)$ is either $1 + f_i$ or f_i , which is nonzero. Since z is normal, so is $\phi(z)$. But $A_n(\mathbb{k})$ is simple, which implies that $\phi(z)$ is a scalar. In this case, the only possibility is $z_i = t$ for all i . \square

Lemma 5.5. *Let A be the non-PI Sklyanin algebra of global dimension at least 3. Then there is no finite set of nonzero elements $\{z_1, \dots, z_w\}$ in degree one such that the product $z_1 z_2 \cdots z_w$ is a normal element in A .*

Proof. This was basically proved in [KKZ1, Corollary 6.]. For completeness we give a proof here. Let n be the global dimension of the Sklyanin algebra A .

Associated to A there is a triple (E, σ, \mathcal{L}) where $E \subseteq \mathbb{P}^{n-1}$ is an elliptic curve of degree n , \mathcal{L} is an invertible line bundle over E of degree n and σ is an automorphism of E induced by a translation. Basic properties of A can be found in [ATV] for $n = 3$, [SS] for $n = 4$, and [TV] for $n \geq 5$. Associated to (E, σ, \mathcal{L}) one can construct the twisted homogeneous coordinate ring, denoted by $B(E, \sigma, \mathcal{L})$. Then there is a canonical surjection

$$\phi : A \rightarrow B(E, \sigma, \mathcal{L}) =: B$$

such that ϕ becomes an isomorphism when restricted to the degree one piece. This statement was proved in [ATV, Section 6] for $n = 3$, [SS, Lemma 3.3] for $n = 4$, and [TV, (4.3)] for $n \geq 5$. If A is non-PI, then σ has infinite order. Hence B is so-called *projectively simple* by [RRZ1], which means that any proper factor ring of B is finite dimensional. Also note that the GK-dimension of B is 2.

Suppose that there are nonzero elements z_1, \dots, z_w in A of degree 1, such that $x := z_1 z_2 \cdots z_w$ is normal. Let

$$x' = \phi(x) = \phi(z_1) \cdots \phi(z_w) \in B.$$

Since ϕ is an isomorphism in degree 1, each $\phi(z_i) \neq 0$. Now a basic property of B is that it is a domain. Hence $x' \neq 0$. Since x is normal, so is x' . Therefore $B/(x')$ is an infinite dimensional proper factor ring of B , which contradicts the fact that B is projectively simple. \square

Now we are ready to prove Theorem 0.9.

Theorem 5.6. *Let \mathbb{k} be an algebraically closed field of characteristic zero.*

- (1) *Let A be the homogenization of the universal enveloping algebra of a finite dimensional semisimple Lie algebra $H(\mathfrak{g})$. For every finite group G , $A^{\mathbb{k}^G}$ is not AS regular. As a consequence, A is rigid with respect to group coactions.*
- (2) *Let A be the Rees ring of the Weyl algebra $A_n(\mathbb{k})$ with respect to the standard filtration. If G is a finite group such that $A^{\mathbb{k}^G}$ is AS regular, then $G = \mathbb{Z}/(2)$ and $A^{\mathbb{k}^G} \not\cong A$. As a consequence, A is rigid with respect to group coactions.*
- (3) *Let A be the non-PI Sklyanin algebras of global dimension at least 3. For every finite group G , $A^{\mathbb{k}^G}$ is not AS regular. As a consequence, A is rigid with respect to group coactions.*

Proof. (1) Assume to the contrary that G is a nontrivial finite group such that $A^{\mathbb{k}^G}$ is AS regular. By Lemma 5.1 and (E5.1.1), $f_m = c f_{g_1} \cdots f_{g_w}$ where $g_i \in \mathfrak{R}$. Then f_{g_i} is of degree 1. By Theorem 0.5(1), f_m is a normal element. By Lemma 5.3, each f_{g_i} is a scalar multiple of t . This implies that all g_i are the same and G is generated

by g_1 , whence abelian. In this case the Hopf algebra \mathbb{k}^G is cocommutative. Since \mathbb{k} is algebraically closed, H is isomorphic to a group algebra $\mathbb{k}G_0$ for some group G_0 (in fact $G_0 \cong G$ is cyclic). By [KKZ1, Theorem 2.4], G_0 contains a quasi-reflection. This contradicts [KKZ1, Lemma 6.5(d)]. Therefore the assertion follows, and the consequence is obvious.

(2) Let G be a nontrivial finite group such that $A^{\mathbb{k}^G}$ is AS regular. By the proof of part (1), there is a nontrivial cyclic group G_0 such that $A^{\mathbb{k}^G} = A^{G_0}$ with some natural action of G_0 on A and $\mathbb{k}^G \cong \mathbb{k}G_0$. By [KKZ1, Proposition 6.7], $G_0 = \mathbb{Z}/(2)$ and by [KKZ1, Corollary 6.8], $A^{G_0} \not\cong A$. Note that $G \cong G_0 = \mathbb{Z}/(2)$. So the assertion follows, and the consequence is obvious.

(3) The assertion follow from Lemmas 5.1 and 5.5. The consequence is obvious. \square

Acknowledgments. The authors would like to thank Monty McGovern for his proof of Lemma 5.2(2). E. Kirkman was partially supported by grant #208314 from the Simons Foundation. J.J. Zhang was partially supported by the US National Science Foundation (grant No DMS 1402863).

REFERENCES

- [AP] J. Alev and P. Polo, A rigidity theorem for finite group actions on enveloping algebras of semisimple Lie algebras, *Adv. Math.* **111** (1995), no. 2, 208–226.
- [Al] A. Aljouiee, On weak crossed products, Frobenius algebras, and the weak Bruhat ordering, *J. Algebra* **287** (2005), no. 1, 88–102.
- [AS] M. Artin and W. F. Schelter, Graded algebras of global dimension 3, *Adv. in Math.* **66** (1987), no. 2, 171–216.
- [ATV] M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, “The Grothendieck Festschrift,” Vol. I, ed. P. Cartier et al., Birkhäuser Boston 1990, 33–85.
- [AZ] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.* **109** (1994), 228–287.
- [Ba] Y. Bazlov, Nichols-Woronowicz algebra model for Schubert calculus on Coxeter groups, *J. Algebra* **297** (2006), no. 2, 372–399.
- [BB] Y. Bazlov and A. Berenstein, Mystic reflection groups, *SIGMA Symmetry Integrability Geom. Methods Appl.* **10** (2014), Paper 040, 11 pp.
- [BjB] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, **231**, Springer, New York, 2005.
- [BRR] F. Brenti, V. Reiner and Y. Roichman, Alternating subgroups of Coxeter groups, *J. Combin. Theory Ser. A* **115** (2008), no. 5, 845–877.
- [BZ] K.A. Brown and J.J. Zhang, Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras, *J. Algebra*, **320** (2008), no. 5, 1814–1850.
- [CWZ] K. Chan, C. Walton and J.J. Zhang, Hopf actions and Nakayama automorphisms, *J. Algebra* **409** (2014), 26–53.
- [CKWZ] K. Chan, E. Kirkman, C. Walton and J.J. Zhang, Quantum binary polyhedral groups and their actions on quantum planes, ArXiv 1303.7203, to appear in *J. Reine Angew. Math.* (Crelles Journal).
- [CM] M. Cohen and S. Montgomery, Group-graded rings, smash products, and group actions, *Trans. Amer. Math. Soc.* **282** (1984), no. 1, 237–258.
- [EW1] P. Etingof and C. Walton, Semisimple Hopf actions on commutative domains, *Adv. Math.* **251** (2014), 47–61.
- [EW2] P. Etingof and C. Walton, Finite dimensional Hopf actions on algebraic quantizations, arXiv:1605.00560.
- [FS] S. Fomin and R.P. Stanley, Schubert Polynomials and the Nilcoxeter Algebra, *Adv. Math.* **103** (1994), 196–207.

- [He] H. Helfgott, Growth in groups: ideas and perspectives, *Bull. Amer. Math. Soc. (N.S.)* **52** (2015), no. 3, 357–413
- [KM] A.N. Kirillov and T. Maeno, Nichols-Woronowicz model of coinvariant algebra of complex reflection groups, *J. Pure Appl. Algebra* **214** (2010), no. 4, 402–409.
- [Ki] E. Kirkman, Invariant theory of Artin-Schelter regular algebras: a survey, preprint, (2015), arXiv:1506.06121v1.
- [KKZ1] E. Kirkman, J. Kuzmanovich, and J.J. Zhang, Rigidity of graded regular algebras, *Trans. Amer. Math. Soc.* **360** (2008), no. 12, 6331–6369.
- [KKZ2] E. Kirkman, J. Kuzmanovich and J.J. Zhang, A Shephard-Todd-Chevalley theorem for noncommutative regular algebras, *Algebr. Represent. Theory* **13** (2010), no. 2, 127–158
- [KKZ3] E. Kirkman, J. Kuzmanovich and J.J. Zhang, Gorenstein subrings of invariants under Hopf algebra actions, *J. Algebra* **322** (2009), no. 10, 3640–3669.
- [KKZ4] E. Kirkman, J. Kuzmanovich, and J.J. Zhang, Noncommutative complete intersections, *J. Algebra* **429** (2015), 253–286.
- [KKZ5] E. Kirkman, J. Kuzmanovich, and J.J. Zhang, Dual reflection groups: Examples, in preparation.
- [LMZ] J.-F. Lü, X.-F. Mao and J.J. Zhang, Nakayama automorphism and applications, *Trans. Amer. Math. Soc.*, (to appear), preprint arXiv:1408.5761.
- [MS] A. Milinski and H.-J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, *New trends in Hopf algebra theory (La Falda, 1999)*, 215236, *Contemp. Math.*, **267**, Amer. Math. Soc., Providence, RI, 2000.
- [PP] A. Polishchuk and L. Positselski, Quadratic algebras, *University Lecture Series*, **37**, American Mathematical Society, Providence, RI, 2005.
- [RRZ1] Z. Reichstein, D. Rogalski and J.J. Zhang, Projectively simple rings, *Adv. in Math.* **203** (2006), 365–407.
- [RRZ2] M. Reyes, D. Rogalski and J.J. Zhang, Skew Calabi-Yau algebras and homological identities, *Adv. Math.* **264** (2014), 308–354.
- [RRZ3] M. Reyes, D. Rogalski and J.J. Zhang, Skew Calabi-Yau triangulated categories and Frobenius Ext-algebras, *Trans. Amer. Math. Soc.*, (to appear), preprint (2014) arXiv:1408.0536.
- [Ro] A. Rotbart, Generator sets for the alternating group, *Sém. Lothar. Combin.* **65** (2010/12), Art. B65b, 16 pp.
- [ST] G.C. Shephard and J.A. Todd, Finite unitary reflection groups, *Canadian J. Math.* **6**, (1954), 274–304.
- [SS] S. P. Smith and J. T. Stafford, Regularity of the four-dimensional Sklyanin algebra, *Compositio Math.* **83** (1992), no. 3, 259–289.
- [SV] D. Stefan and C. Vay, The cohomology ring of the 12-dimensional Fomin-Kirillov algebra, *Adv. Math.* **291** (2016), 584620.
- [TV] J. Tate and M. Van den Bergh, Homological properties of Sklyanin algebras, *Invent. Math.* **124** (1996), no. 1-3, 619–647.
- [VdB] M. Van den Bergh, Existence theorems for dualizing complexes over noncommutative graded and filtered rings, *J. Algebra* **195** (1997), 662–679.
- [YZ] A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, *J. Algebra* **213** (1999), no. 1, 1–51.

KIRKMAN: DEPARTMENT OF MATHEMATICS, P. O. BOX 7388, WAKE FOREST UNIVERSITY, WINSTON-SALEM, NC 27109

E-mail address: kirkman@wfu.edu

KUZMANOVICH: DEPARTMENT OF MATHEMATICS, P. O. BOX 7388, WAKE FOREST UNIVERSITY, WINSTON-SALEM, NC 27109

E-mail address: kuz@wfu.edu

ZHANG: DEPARTMENT OF MATHEMATICS, BOX 354350, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195, USA

E-mail address: zhang@math.washington.edu